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# Symmetry and Composition—A Key to the Structure of Physical Logic?

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From considerations about the structure of arbitrary physical theories, conditions are derived for physical logic, mainly concerning symmetry and tensor products. The quantum measurement is discussed as a chief example.

We ask for a justification of quantum logic or, to put it more generally, of a logic of physics. We can learn a lot about this logic by asking: "What must a structure look like in order to be recognized as a physical theory?" This is very near the Kantian search for a justification *a priori* that I have dealt with elsewhere (Drieschner, 1992, 1993; see also Drieschner, 1979). The fundamental definition of a physical theory, according to those considerations, is: *a set of rules for the derivation of empirically testable predictions from present conditions.*

## 1. PREDICTIONS

To begin with: If we want an empirical test, we have to have something that can be tested: possible outcomes of an experiment, something that is called "observable," "test," "partition of unity," "$n$-fold alternatively." It spans a Boolean lattice, as in classical logic. We ask how Boolean lattices can be connected among each other if they are not compatible, i.e., if they cannot be embedded into one single Boolean lattice. In order to answer this question we have to consider the special temporal structure of laws of nature (Drieschner, 1992): A law of nature has the form: "Under (specified) circumstances a certain measurement will have (specified) results"; it is a general rule for the derivation of specific *predictions.* It is necessary that a theory

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give predictions, or else there could not be an empirical *test*: It tests whether a prediction comes true.

In many discussions this predictive structure of physics has not been considered seriously enough. We shall see how important it is. It is clear that the elements of several (physical) Boolean algebras always form a poset or an orthoalgebra. From the predictive structure we can even conclude that they form a lattice: Let $a$ and $b$ be incompatible elements such that they cannot be true at the same time. But if they are predictions, the conjunction "$a \wedge b$" has a good sense: We have to use the proposition "$a$ is necessary" or "$p(a) = 1$" [where $p(a)$ is the probability of $a$] instead of "$a$ is true." In that case the conjunction means that $a$ is necessary as well as $b$ is necessary; the experimenter may choose which one he wants to test. And the *prediction* is, whichever proposition he tests, it will come out true. Possibly this prediction will never really be made (namely if $a \wedge b = 0$ in the lattice), but still it has formally a good meaning (Drieschner, 1993; see also Drieschner, 1979). This gives us immediately the Jauch–Piron property for our lattice: $p(a) = 1$, $p(b) = 1 \Rightarrow p(a \wedge b) = 1$. This consequence, by the way, answers also Holland's (1995) question concerning his axiom A2: Together with "negation" (below) our considerations give a physical justification for Holland's axiom.

## 2. PROBABILITY FUNCTION

We have already used a probability function in order to express the necessity of prediction $a$. It can be shown that probability is the most general empirically testable prediction we can think of (Drieschner, 1993; see also Drieschner, 1979): If I cannot predict "yes" or "no," I must at least be able to predict the relative frequency of "yes" and "no" outcomes, i.e., to give a probability. I cannot give here the details of the argument; the conclusion is that *every* physical theory must give probabilities, possibly (in the "classical", degenerate case) only probabilities 0 and 1. This means that probability distributions ("states") are important for our reasoning about the structure of the physical logic.

We conclude from the foregoing that every set of probability measures on our lattice is *strong*: The order relation of the lattice is logical implication. The elements of the lattice are predictions. Now "prediction $a$ implies prediction $b$" means: "If $a$ is necessary, then $b$ is necessary." Thus we have the equivalence

$$[a \Rightarrow b] \Leftrightarrow [p(a) = 1 \Rightarrow p(b) = 1]$$

by the structure of $L$.

In a similar way we conclude from the structure of *negation*: Every empirical test of a physical prediction tests whether this prediction is true or false. If it is false, its negative is true. Negation is represented by orthocomplementation, as can be easily seen: The definition of orthocomplementation gives exactly the rules that characterize logical negation. This means that $\mathrm{p}(a) = 0$ implies $\mathrm{p}(a') = 1$; every state is *consistent.*

## 3. SYMMETRY

The description of every physical object is such that we can distinguish between the internal relations between its properties on the one hand, and external criteria that distinguish such properties individually, on the other. It is, e.g., an old "topos" of philosophy (Leibniz, 1694) that nothing in our experience would change if all objects were translated, say, 2 miles north: In order to distinguish the location of an object from other locations we have to refer to its relations to other objects. Ernst Mach (1933) emphasized this necessity so far as to even postulate the relativity of rotation ("Mach's principle").

More generally, any observable is defined only by a measuring instrument and a measuring process *outside* the object that is measured. Thus the *internal* structure of an object does not distinguish between (atomic) properties; it is invariant under transformations that leave the relations between its properties intact.

We can argue for that symmetry in a still more general way: A law of nature describes changes of objects that occur at any period of time in the same way: The sun rises every morning, the Stern–Gerlach experiment works the same way any time we make it. This implies that temporal development is described by a group representation of $R^+$ ("time") in the automorphisms of the "logic" of the object—in quantum mechanics by the one-parameter unitary Schrödinger representations $\exp(it\mathcal{H})$. Thus, consider a set of properties the same object can have at different times; this set has to admit the kind of automorphism group mentioned above, i.e., it has to have that symmetry.

Apparently this postulate will exclude some of the counterexamples that are usually given in the discussion about "physical" logic.

## 4. COUPLING OF SYSTEMS

It must be possible, in any physical theory, to describe two objects at a time, i.e., to treat them as one object. This must be possible even if (or, as long as) those objects are entirely independent.

This implies the structure of the tensor product: Since each one of the objects can be described independently, their joint structure has to admit the corresponding bimorphisms as well as the product rule of probability theory.

We are considering a general physical theory, i.e., a theory that describes the changes in time of arbitrary physical objects (a general "mechanics," like quantum mechanics). The generality of that theory means that its "logic" is of a certain category, independently of the size of the objects described. Thus, for its "true" logic we need only consider categories that are closed under the formation of tensor products. By this argument, as far as I can see, from the logics considered so far only the following remain:

1. Unital orthoalgebras (Foulis *et al.*, 1992).
2. Difference posets that have a sufficient system of states (e.g., Dvurečenskij, 1995).
3. Hilbert space lattices, as is well known.

It seems to be a nice program to investigate the combination of all the conditions mentioned so far and see which structures comply with all of them. It seems quite possible that the conditions of Solèr's (1995) theorem are met, such that only Hilbert space lattices remain; if not, it will be quite interesting to name all structures that are possible under those conditions.

## 5. PROBABILISTIC INDEPENDENCE VERSUS NONINFLUENCE

In probability theory we call two probability distributions $p(a_i)$, $p(b_j)$ independent iff the product rule holds: $p(a_i \wedge b_j) = p(a_i), p(b_j)$, for all $i$, $j$. Two physical systems $A$ and $B$, on the other hand, are independent iff the "motion" (change of state) of $A$ is the same whatever state $B$ is in, and vice versa. The two concepts of independence are not the same, although they may be closely connected. Take, e.g., the description of the measuring process; we distinguish three phases: In the first phase, *before* the measurement, object and apparatus are physically independent, and the probability distributions are independent as well; in the *second* phase, the measurement interaction between the two takes place; and in the *third* phase, after this interaction, the object and apparatus are physically independent again, but their probability distributions are no longer independent: If the apparatus has property $\phi_i$, then the probability must be 100% that the object has the corresponding property $\varphi_i$.

Hence it is important, if we speak of independence, to be careful with this distinction.

## 6. QUANTUM MEASUREMENT

The problem with quantum measurement is that its description contains a contradiction in its very foundations: If we consider quantum mechanics as a fundamental physical theory, we must admit that the instrument is described quantum mechanically, as well as the object. Let us first look at the very simple example of an object with a two-dimensional "logic" (Hilbert space), admitting an observable $Q$ with eigenstates $\varphi_1$ and $\varphi_2$.

We can discuss the structure of the process of measurement in the framework of a rather general physical logic: An instrument that measures $Q$ admits three "pointer" states: $\phi_0$ for the "ready" state, $\phi_1$ and $\phi_2$ for the states indicating object states $\varphi_1$ or $\varphi_2$, respectively. The instrument measures $Q$; this *means* that, if the object is in eigenstate $\varphi_1$ in the beginning, then the interaction of the measurement leads to the final state $\varphi_1 \times \phi_1$ in the tensor product logic; if the object is in eigenstate $\varphi_2$, the result of the measurement interaction is $\varphi_2 \times \phi_2$. There must be an automorphism $U$, according to the generalized Schrödinger equation, that does the corresponding transformation:

$$U(\varphi_1 \times \phi_0) = \varphi_1 \times \phi_1$$

$$U(\varphi_2 \times \phi_0) = \varphi_2 \times \phi_2$$

These two equations define the automorphism $U$ completely.

Let the measured object be in an arbitrary atom ("pure state") $\varphi_x$ of the logic. The atom $\varphi_x \times \phi_0$ of the tensor product space, as above, is transformed by $U$ into $U(\varphi_x \times \phi_0)$, which is incompatible with both $\varphi_1 \times \phi_1$ and $\varphi_2 \times \phi_2$ if $\varphi_x \neq \varphi_1$ and $\varphi_x \neq \varphi_2$.

[In Hilbert space theory this means: The general initial pure state $\varphi_x$ of the measured object can be written as

$$\varphi_x = \alpha \cdot \varphi_1 + \beta \cdot \varphi_2$$

with complex coefficients $\alpha$ and $\beta$. The automorphism $U$ (a unitary transformation) transforms it into

$$U(\varphi_x \times \phi_0) = \alpha \cdot \varphi_1 \times \phi_1 + \beta \cdot \varphi_2 \times \phi_2 \tag{1}$$

which is again a pure state.]

But let us now look at the *result* of the measurement; here the contradiction arises: The instrument will either show result 1 or result 2, being in the final state $\phi_1$ or $\phi_2$, respectively. These states of the instrument indicate that the object is in state $\varphi_1$ or $\varphi_2$, respectively (we assume, for simplicity, an ideal measurement of the first kind). Consider, as above, the initial pure state $\varphi_x$ of the object, with probability $p_1$ ($p_2$) for finding result 1 (result 2), with $p_1 + p_2 = 1$. Now we use the tensor product for describing the compound object after the measurement. Its description is $\varphi_1 \times \phi_1$ with probability $p_1$, and $\varphi_2 \times \phi_2$ with probability $p_2$—which is a general formulation of "mixture."

This description is apparently different from, even incompatible with, the state $U(\varphi_x \times \phi_0)$ described above (1).

[In Hilbert space, again, we have the well-known difference between the corresponding density operators $W$ and $W'$ (where $P_\varphi$ is the projector onto $\varphi$):

The result of the automorphism $U$ is [cf. (1)].

$$W = P_{(\alpha \cdot \varphi_1 \times \phi_1 + \beta \cdot \varphi_2 \times \phi_2)}$$

The "mixture," on the other hand, is

$$W' = |\alpha|^2 \cdot P_{(\phi_1 \times \phi_1)} + |\beta|^2 \cdot P_{(\varphi_2 \times \phi_2)}$$

where $W \neq W'$ if $\alpha, \beta > 0$.]

This is the simplest possible formulation of the problem, but a long discussion[2] shows that nothing changes fundamentally if we go over to observables with many possible outcomes, replace the pure states of the object or apparatus with mixtures, or consider measurements of other kinds. Our analysis shows that the problem persists even in the much more general setting of quantum logic.

## 7. LINEARITY

Wigner as well as Busch *et al.* (1991, Section IV.4.2) conclude from these considerations that the Schrödinger equation ought to be replaced by something nonlinear. But, as I have argued, there are rather strong arguments for the claim that even the most general theory of physics has to describe time translations as a one-parameter group of automorphisms of the quantum logic. This implies the (generalized) Schrödinger equation. And that leaves no room for a nonlinear description of temporal development.

The solution I propose looks rather easy, but it touches on fundamental questions: The two density operators $W$ and $W'$ can be made much alike in a real measurement, even though they can never be exactly equal. It belongs to the fundamental meaning of measurement that we *separate* the object from the instrument. This is always an approximation, but that approximation lies at the foundation of physics; so we just have to accept it (cf. Drieschner, 1979)!

[2] See Süssmann (1958), Wigner (1963), and Busch *et al.* (1991). In this latter book a different formulation of the inconsistency is given (cf. section III.6.2), but it leads to basically the same problem.

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